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Limits and Derivatives

Every time a real-world application gets close to a stable solution, the real-life bounds are applied. A chemical reaction that starts in a beaker and involves two separate chemicals reacting to create a new compound is an example of a limit. The amount of the newly created compound has a limit now that time is getting closer to infinity.

Topic Notes

- *Fundamental of Limits*
- *Derivatives*

FUNDAMENTAL OF LIMITS 1

TOPIC 1

LIMITS OF ALGEBRAIC FUNCTIONS

In this section, we shall discuss various methods to evaluate the limits involving algebraic expressions of the form $\frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are expressions in the variable x and $q(x) \neq 0$.

Direct Substitution Method

We shall now discuss the direct substitution method for the evaluation of limits.

When to Apply

This method is used to evaluate the limits of form $\lim_{x \to a} [f(x)]$, where a lies in the domain of the function $f(x)$ (i.e., $f(a)$ exists).

Evaluate $f(a)$, which is the required limit.

Computing Limits

In the following sections, we shall discuss the algebraic techniques for computing the limits of various functions. While using these techniques, we make use of the following properties of limits:

(i) $\lim_{x \to a} [k] = k$, where k is a constant.

(ii) $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} [f(x)] + \lim_{x \to a} [g(x)]$.

(iii) $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} [f(x)] - \lim_{x \to a} [g(x)]$.

(iv) $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} [f(x)] \lim_{x \to a} [g(x)]$.

(v) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} [f(x)]}{\lim_{x \to a} [g(x)]}$ provided $\lim_{x \to a} [g(x)] \neq 0$.

Let $a \in R$ and let f be a real-valued function in real variable x defined at the points in an open interval containing a except possibly at a. Then, we say that the limit of the function $f(x)$ is a real number L, as x tends to a, if the value of $f(x)$ approaches L as x approaches a. It is denoted by $\lim_{x \to a} f(x)$ and is read as 'limit of $f(x)$ as x tends to a'.

Limits of a Polynomial Function

A function f is said to be a polynomial function, if $f(x)$ is a zero function or if $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$, where a_i's are real number such that $a_n \neq 0$.

Then, limit of polynomial functions is

$$f(x) = \lim_{x \to a} f(x) = \lim_{x \to a} [a_0 + a_1x + a_2x^2 + ... + a_nx^n]$$
$$= a_0 + a_1a + a_2a^2 + ... + a_na^n = f(a)$$

Limits of Rational Functions

A function f is said to be a rational function, if $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomial functions such that, $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)} = \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$

However, if $h(a) = 0$, then there are two cases arise,

(i) $g(a) \neq 0$ (ii) $g(a) = 0$

In the first case, we say that the limit does not exist.

In the second case, we can find a limit.

Rationalisation Method

This is particularly used when either the numerator or denominator or both in value expression easily of square roots and substituting the value of x, the rational expression takes the form $\frac{0}{0}, \frac{\infty}{\infty}$, etc.

Algebraic limits by using standard limits if $n \in \theta$ then

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = xa^{n-1}$$

Trigonometric Functions

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \quad \lim_{x \to 0} \frac{\tan x}{x} = 1$$

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 1, \quad \lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$$

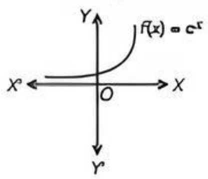
$$\lim_{x \to 0} \frac{\log_e (1 + n)}{x} = 1, \quad \lim_{x \to 0} \frac{e^x - 1}{x} = 1.$$

Limits of Exponential Functions

The great Swiss Mathematician Leonhard Euler (1707-1783) introduced the number e, whose value lies between 2 and 3. This number is useful in defining exponential function.

A function of the form of $f(x) = e^x$ is called exponential function.

The graph of the function is given below:



(i) Domain of $f(x) = (-\infty, \infty)$

(ii) Range of $f(x) = (0, \infty)$

To find the limit of a function involving exponential function, we use the following theorem.

Theorem: $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof: We have an inequality

$$\frac{1}{1+|x|}\le\frac{e^x-1}{x}\le 1+(e-2)|x|,$$

$$x\in[-1,1]-\{0\} \quad \text{...(i)}$$

Here, $\frac{e^x-1}{x}$ is sandwiched between the functions $\frac{1}{1+|x|}$ and $[1+(e-2)|x|]$.

Now, $\lim_{x\to 0}\frac{1}{1+|x|}=\frac{\lim_{x\to 0}1}{\lim_{x\to 0}(1+|x|)}$ [by quotient of limits]

$$=\frac{1}{1+|0|}=\frac{1}{1}=1$$

and $\lim_{x\to 0}[1+(e-2)|x|]=1+(e-2)|0|$

$$=1+(e-2)(0)=1$$

Thus, $\lim_{x\to 0}\frac{1}{1+|x|}=1=\lim_{x\to 0}[1+(e-2x)|x|]$

So, by applying sandwich theorem in eq. (i), we get

$$\therefore \quad \lim_{x\to 0}\frac{e^x-1}{x}=1$$

Hence, proved.

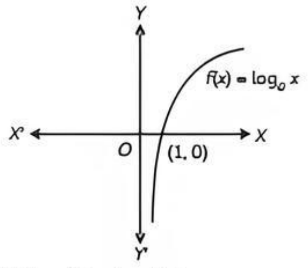
Method to Find the Limit of Exponential Functions

If given function has exponential term, then we convert the given theorem in the form of $\frac{e^x-1}{x}$ and then use the theorem $\lim_{x\to 0}\frac{e^x-1}{x}=1$.

Limits of Logarithmic Functions

The logarithmic function expressed as log k was given by $\log_e x=y$ iff $e^y=x$.

The graph of the function is given below:



(i) Domain of $f(x)=(0,\infty)$ or R^+

(ii) Range of $f(x)=(-\infty,\infty)$ or R

To find the limit of functions involving logarithmic function use the following theorem

Theorem: $\lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$

Proof: Let $\frac{\log_e(1+x)}{x}=y$

Then, $\log_e(1+x)=xy$

$\Rightarrow \quad 1+x=e^{xy}$ $[\because \log x=y\Rightarrow e^y=x]$

$\Rightarrow \quad \frac{e^{xy}-1}{x}=1$

$\Rightarrow \quad \frac{e^{xy}-1}{xy}\,y=1$

On taking limit $xy\to 0$ both sides, we get

$$\lim_{xy\to 0}\frac{e^{xy}-1}{xy}\lim_{x\to 0}y=1$$

$$\Rightarrow \quad \lim_{x\to 0}y=1$$

$$\left[\because \lim_{xy\to 0}\frac{e^{xy}-1}{xy}=1 \text{ and as } x\to 0, \text{ then } xy\to 0\right]$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{\log_e(1+x)}{x}=1$$

Note:

$$\lim_{x\to 0}\frac{\log_e(1-x)}{-x}=1$$

Corollary

I. $\lim_{x\to 0}\frac{\log_e(1-x)}{-x}=1$

II. $\lim_{x\to 0}\frac{a^x-1}{x}=\log a$

Method to Find the Limit of Logarithmic Function

In given function involves logarithmic functions, then we convert the given function in the form of $\frac{\log_e(1+x)}{x}$ and then we the theorem $\lim_{x\to 0}\frac{(1+x)}{x}=1$.

Example 1.1: Evaluate the given limit:

$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$. [NCERT]

Ans. $\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$

$$=\lim_{x\to 0}\frac{(1-2\sin^2 x)-1}{\cos x-1}$$

$[\because \cos 2x=1-2\sin^2 x]$

$$= \lim_{x \to 0} \frac{-2\sin^2 x}{\cos x - 1}$$

$$= \lim_{x \to 0} \frac{-2(1-\cos^2 x)}{\cos x - 1}$$

$[\because \sin^2 x = 1 - \cos^2 x]$

$$= \lim_{x \to 0} \frac{-2(1-\cos^2 x)}{-1(1-\cos x)}$$

$$= \lim_{x \to 0} \frac{2(1^2-\cos^2 x)}{1-\cos x}$$

$$= \lim_{x \to 0} 2(1 + \cos x)$$

On putting $x = 0$, we get

$= 2(1 + \cos 0)$

$= 2(1 + 1)$

$= 2 \times 2$

$= 4$

Example 1.2: Evaluate the given limit:

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}.$$ [NCERT]

Ans. Given, $\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Putting, $x = 3$

$$= \frac{(3)^4 - 81}{2(3)^2 - 5(3) - 3}$$

$$= \frac{81 - 81}{18 - 15 - 3}$$

$$= \frac{0}{0}$$

Since, it is of the $\frac{0}{0}$ form.

We simplify as,

$$\lim_{x \to 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \to 3} \frac{(x^2)^2 - (9)^2}{2x^2 - 6x + x - 3}$$

$[\because a^2 - b^2 = (a - b)(a + b)]$

$$= \lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{2x(x-3) + 1(x-3)}$$

$$= \lim_{x \to 3} \frac{(x^2 - 9)(x^2 + 9)}{2x(x-3) + 1(x-3)}$$

$$= \lim_{x \to 3} \frac{(x-3)(x+3)(x^2 + 9)}{(2x+1)(x-3)}$$

$$= \lim_{x \to 3} \frac{(x+3)(x^2 + 9)}{2x+1}$$

Putting $x = 3$

$$= \frac{(3+3)[(3)^2 + 9]}{2 \times 3 + 1}$$

$$= \frac{6(9+9)}{6+1}$$

$$= \frac{6(18)}{7}$$

$$= \frac{(108)}{7}$$

Example 1.3: Evaluate the given limit: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$. [NCERT]

Ans. Given: $\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

$$= \frac{(1)^{\frac{1}{3}} - 1^1}{(1)^{\frac{1}{6}} - 1}$$

$$= \frac{1-1}{1-1}$$

$$= \frac{0}{0}$$

Since, it is form $\frac{0}{0}$.

We can solve it by using

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$$

Hence,

$$\lim_{z \to 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

$$= \lim_{z \to 1} z^{\frac{1}{3}} - 1 + \lim_{z \to 1} z^{\frac{1}{6}} - 1$$

$$= \lim_{z \to 1} [z^{\frac{1}{3}} - (1)^{\frac{1}{3}}] + \lim_{z \to 1} [z^{\frac{1}{6}} - (1)^{\frac{1}{6}}]$$

Multiplying and dividing by $z - 1$

$$= \lim_{z \to 1} \frac{z^{\frac{1}{3}} - (1)^{\frac{1}{3}}}{z - 1} + \lim_{z \to 1} \frac{z^{\frac{1}{6}} - (1)^{\frac{1}{6}}}{z - 1}$$

$$\lim_{z \to 1} \frac{z^{\frac{1}{3}} - (1)^{\frac{1}{3}}}{z-1} \qquad \lim_{z \to 1} \frac{z^{\frac{1}{6}} - (1)^{\frac{1}{6}}}{z-1}$$

$$= \frac{1}{3}(1)^{\frac{1}{3}-1} \qquad = \frac{1}{6}(1)^{\frac{1}{6}-1}$$

$$\left[\because \lim_{z \to 1} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{1}{3} \times 1 \qquad = \frac{1}{6} \times 1$$

$$= \frac{1}{3} \qquad = \frac{1}{6}$$

Hence, our equation becomes

$$= \lim_{z \to 1} \frac{z^{\frac{1}{3}} - (1)^{\frac{1}{3}}}{z-1} + \lim_{z \to 1} \frac{z^{\frac{1}{6}} - (1)^{\frac{1}{6}}}{z-1}$$

$$= \frac{1}{3} + \frac{1}{6}$$

$$= \frac{1}{3} \times \frac{6}{1}$$

$$= 2$$

Example 1.4: Evaluate the given limit:

$$\lim_{x \to 0} \frac{\sin ax}{\sin bx}, a, b \neq 0. \qquad \text{[NCERT]}$$

Ans. Given, $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

$$\lim_{x \to 0} \sin ax \lim_{x \to 0} \frac{1}{\sin bx}$$

Multiplying and dividing by ax

$$\lim_{x \to 0} \frac{\sin ax}{ax} \times \lim_{x \to 0} \frac{1}{\sin bx} ax$$

$$= 1 \times \lim_{x \to 0} \frac{ax}{\sin bx} \qquad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \lim_{x \to 0} \frac{ax}{\sin bx}$$

Replacing x by ax

$$\left[\lim_{x \to 0} \frac{\sin ax}{ax} = 1\right]$$

Multiplying and dividing by bx

$$= \lim_{x \to 0} \frac{ax}{\sin bx} \times \frac{bx}{bx}$$

$$= \lim_{x \to 0} \frac{bx}{\sin bx} \times \frac{a}{b}$$

$$= \frac{a}{b} \times \lim_{x \to 0} \frac{bx}{\sin bx}$$

$$= \frac{a}{b} \div \lim_{x \to 0} \frac{\sin bx}{bx}$$

[using $\lim_{x \to 0} \frac{\sin x}{x} = 1$ replacing x by ax

$$\lim_{x \to 0} \frac{\sin ax}{bx} = 1]$$

$$= \frac{a}{b} \div 1$$

$$= \frac{a}{b}$$

Example 1.5: Evaluate $\lim_{x \to 0} f(x)$, where

$$f(x) = \begin{cases} \frac{|x|}{x}, x \neq 0 \\ 0, x = 0 \end{cases} \qquad \text{[NCERT]}$$

Ans. Finding limit at $x = 0$

LHL at x 0

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} f(-h)$$

$$= \lim_{h \to 0} \frac{|-h|}{-h}$$

$$= \lim_{h \to 0} \frac{h}{-h}$$

$$= \lim_{h \to 0} -1$$

$$= -1$$

RHL at $x \to 0$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} f(0+h)$$

$$= \lim_{h \to 0} f(h)$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

$$= \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1$$

$$= 1$$

Since Left Hand Limit (LHL) ≠ Right Hand Limit (RHL)

$\therefore \lim_{h \to 0} f(x)$ doesn't exist.

Example 1.6: Suppose $f(x) = \begin{cases} a+bx, x<1 \\ 4, x=1 \\ b-ax, x>1 \end{cases}$ and if $\lim_{x \to 1} f(x) = f(1)$, what are possible values of a and b? [NCERT]

Ans. Here, limit exist at $x \to 1$

i.e., LHL = RHL = $f(1) = 4$ ...(i)

LHL at $x \to 1$

$$\lim_{x \to 1} f(x) = \lim_{x \to 1} f(1-h)$$
$$= \lim_{h \to 0} a + b(1-h)$$
$$= a + b(1-0)$$
$$= a + b \quad \text{...(ii)}$$

From (i) and (ii)

$$a + b = 4$$

RHL at $x \to 1$

$$\lim_{x \to 1} f(x) = \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} b - a(1+h)$$
$$= b - a(1+0)$$
$$= b - a \quad \text{...(iii)}$$

From (i) and (iii)

$$b - a = 4$$

Adding both

$$a + b + b - a = 4 + 4$$
$$2b = 8$$
$$b = 4$$

Also,

$$a + b = 4$$
$$a + 4 = 4$$
$$a = 0$$

Thus, $a = 0$ and $b = 4$.

Example 1.7: If $f(x) = \begin{cases} mx^2 + n, x < 0 \\ nx + m, 0 \le x \le 1 \\ nx^3 + m, x > 1 \end{cases}$, for what integers m and n does $\lim_{x \to 0} f(x)$ and $\lim_{x \to 1} f(x)$ exist? [NCERT]

Ans. Given limit exists at $x = 0$ and $x = 1$.

At, $x = 0$

Limit exists at x = 0 if

Left-hand limit = right-hand limit

$$f(x) = \begin{cases} mx^2 + n, x < 0 \\ nx + m \ \ 0 \le x \le 1 \\ nx^3 + m, x > 1 \end{cases}$$

LHL at $x \to 0$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0-h)$$
$$= \lim_{h \to 0} f(-h)$$
$$= \lim_{h \to 0} m(-h)^2 + n$$
$$= m(0)^2 + n$$
$$= n$$

RHL at $x \to 0$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(0+h)$$
$$= \lim_{h \to 0} f(h)$$
$$= \lim_{h \to 0} nh + m$$
$$= n(0) + m$$
$$= m$$

Since for limit to exist, LHL = RHL

$\therefore \quad m = n$

So, $\lim_{x \to 0} f(x)$ exists if $m = n$.

Now, a limit exists at $x = 1$.

Therefore,

Left-hand limit = Right-hand limit

$$f(x) = \begin{cases} mx^2 + n, x < 0 \\ nx + m \ \ 0 \le x \le 1 \\ nx^3 + m, x > 1 \end{cases}$$

LHL at $x \to 1$

$$\lim_{x \to 0} f(x) = \lim_{h \to 0} f(1-h)$$
$$= \lim_{h \to 0} n(1-h) + m$$
$$= n(1-0) + m$$
$$= n + m$$

RHL at $x \to 1$

$$\lim_{h \to 0} f(x) = \lim_{h \to 0} f(1+h)$$
$$= \lim_{h \to 0} n(1+h)^3 + m$$
$$= n(1-0)^3 + m$$
$$= n + m$$

Since, for limit to exist, LHL = RHL

$$m + n = m + n$$

which is always true.

So, $\lim_{x \to 1} f(x)$ exists at all integral values of m and n.

Example 1.8: Evaluate the given limit:

$$\lim_{x \to 1} (\operatorname{cosec} x - \cot x).$$ [NCERT]

Ans. $\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$

$$= \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right)$$

$$\left[\because \ \operatorname{cosec} \theta = \frac{1}{\sin \theta}, \cot \theta = \frac{\cos \theta}{\sin \theta} \right]$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin x}$$

Putting $x = 0$

$$= \frac{1-\cos 0}{\sin 0}$$

$$= \frac{1-1}{0}$$

$$= \frac{0}{0}$$

Thus, we proceed as under

$$\lim_{x \to 0} (\operatorname{cosec} x - \cot x)$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{\sin x}$$

$$= \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \lim_{x \to 0} \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}$$

$$= \lim_{x \to 0} \left(\tan \frac{x}{2}\right)$$

$$= 0$$

Example 1.9: Evaluate the given limit: $\lim_{x \to 0} x \sec x$.

[NCERT]

Ans. Given, $\lim_{x \to 0} x \sec x$

$$= \lim_{x \to 0} x \cdot \frac{1}{\cos x} \qquad \left[\because \sec\theta = \frac{1}{\cos\theta}\right]$$

$$= \lim_{x \to 0} \frac{x}{\cos x}$$

Putting $x = 0$

$$= \frac{0}{\cos x}$$

$$= \frac{0}{1} \qquad (\cos 0 = 1)$$

$$= 0$$

Example 1.10: Case Based:

A class teacher Shinu Sharma of Class XI writes

Let $f : R \to [0, \infty)$ be a function defined as $f(x) = |x|$ and $g(x) = f(x + 1) + f(x - 1)$, $\forall \in R$. and asked students to answer the following question based on the above information

(A) Find the value of $g(x)$.

(B) If $\lim_{x \to -1} g(x) = a$, then the value of a is:

(a) 0 (b) −2

(c) 2 (d) does not exist

(C) Find the value of $\lim_{x \to 1^+} g(x)$.

(D) The value of $\lim_{x \to 1^-} g(x)$ is:

(a) 0 (b) 2

(c) −2 (d) does not exist

(E) Assertion (A): $\lim_{x \to 3} \frac{3^{2+x} - 9}{x}$ is equal to $9 \log 2$.

Reason (R): $\lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x}$ is equal to $\log a$.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

Ans. (A) Given $f(x) = |x|$ and $g(x) = f(x + 1) + f(x - 1)$

So,

$$g(x) = |x + 1| + |x - 1|$$

$$g(x) = \begin{cases} -(x+1)-(x-1), & \text{if } x < -1 \\ (x+1)-(x-1), & \text{if } -1 \le x < 1 \\ (x+1)+(x-1), & \text{if } x \ge 1 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} -2x & \text{if } x < -1 \\ 2 & \text{if } -1 \le x < 1 \\ 2x & \text{if } x \ge 1 \end{cases}$$

(B) *(c)* 2

Explanation: For $\lim_{x \to -1} g(x)$ to exist

$$\lim_{x \to -1^-} g(x) = \lim_{x \to -1^+} g(x)$$

$$\Rightarrow \quad \lim_{x \to -1^-} (-2x) = \lim_{x \to -1^+} 2(x)$$

$$\Rightarrow \quad 2 = 2$$

$\lim_{x \to -1} g(x)$ exists and equal to $2 \Rightarrow a = 2$

(C) $\lim_{x \to 1^+} g(x) = \lim_{x \to 1^+} 2x$

$= 2$

(D) *(b)* 2

Explanation: $\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} 2 = 2$

(E) *(d)* *(A) is false but (R) is true.*

Explanation: Given,

$$\text{limit} = \lim_{x \to 0} \frac{3^{2+x} - 9}{x}$$

$$= \lim_{x \to 0} \frac{3^2(3^x - 1)}{x} = 9 \log 3$$

Now consider,

$$\text{limit} = \lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x}$$

Let, $y = \sin x$

Then, $y \to 0$ as $x \to 0$

$$\therefore \quad \lim_{x \to 0} \frac{a^{\sin x} - 1}{\sin x} = \lim_{y \to 0} \frac{a^y - 1}{y} = \log a$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The value of $\lim_{x \to 3} (4x^3 - 2x^2 - x + 1)$ is:

(a) 40 (b) 20

(c) 38 (d) 88

Ans. *(d)* 88

Explanation:

We have $\lim_{x \to 3} (4x^3 - 2x^2 - x + 1)$

On putting, $x = 3$, we get

$$= 4(3)^3 - 2(3)^2 - 3 + 1$$

$$= 4 \times 27 - 2 \times 9 - 3 + 1$$

$$= 108 - 18 - 2$$

$$= 90 - 2$$

$$= 88$$

2. If $f(x) = \begin{cases} x^2 + 1, & x \geq 1 \\ 3x - 1, & x < 1 \end{cases}$, then the value of $\lim_{x \to 1} f(x)$ is:

(a) 2 (b) –2

(c) 1 (d) –1

[Delhi Gov. Term-1 SQP 2021]

Ans. *(a)* 2

Explanation: Given,

$$f(x) = \begin{cases} x^2 + 1, & x \geq 1 \\ 3x - 1, & x < 1 \end{cases}$$

L.H.L. = R.H.L. = $f(1)$

$$\text{L.H.L.} = \lim_{x \to 1^-} x^2 - 1$$

$$= 3 - 1 = 2$$

$$\text{R.H.L.} = \lim_{x \to 1^+} x^2 + 1$$

$$= 1 + 1$$

$$= 2$$

$$\therefore \quad \lim_{x \to 1} x^2 + 1 = 1 + 1 = 2$$

3. The value of $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$ is:

(a) 10 (b) 11

(c) 12 (d) 13

Ans. *(c)* 12

Explanation:

We have, $\lim_{x \to 2} \frac{x^3 - 8}{x - 2}$

Now, $\frac{x^3 - 2^3}{x - 2} = \frac{(x-2)(x^2 + 2^2 + 2x)}{x - 2}$

$$= x^2 + 4 + 2x$$

$$= 2^2 + 4 + 4$$

$$= 12$$

4. The value of x $\lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1}$ is:

(a) 1 (b) 2
(c) 3 (d) 4

Ans. *(b) 2*

Explanation:
We have,

$$\lim_{x \to \frac{1}{2}} \frac{4x^2-1}{2x-1} = \frac{(2x)^2-1^2}{2x-1}$$

$$= \frac{(2x-1)(2x+1)}{2x-1}$$

$$= 2x + 1$$

$$= 2 \times \frac{1}{2} + 1$$

$$= 2$$

5. The value of $\lim_{x \to 5}\left(\frac{x^2-25}{x-5}\right)$ is:

(a) 0 (b) ∞
(c) 10 (d) 5 [Diksha]

Ans. *(c) 10*

Explanation: $\lim_{x \to 5}\left(\frac{x^2-25}{x-5}\right)$

$$= \lim_{x \to 5} \frac{(x+5)(x-5)}{x-5}$$

$$= \lim_{x \to 5} x + 5 = 5 + 5 = 10$$

6. If $\lim_{x \to 2} \frac{x^n-2^n}{x-2} = 80$ then n is:

(a) 1 (b) 3
(c) 5 (d) 7

Ans. *(c) 5*

Explanation: We have $\lim_{x \to 2} \frac{x^n-2^n}{x-2} = 80$

Using $\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}$

So,
$$n.2^{(n-1)} = 80$$
$$n.2^{(n-1)} = 5 \times 16$$
$$n.2^{n-1} = 5 \times 2^{5-1}$$
$$n = 5$$

7. The value of $\lim_{x \to 0} \frac{(x+2)^{1/3}-2^{1/3}}{x}$ is:

(a) $\frac{1}{3(2)^{3/2}}$ (b) $\frac{1}{3(2)^{2/3}}$

(c) $\frac{1}{2(3)^{2/3}}$ (d) $\frac{1}{3\sqrt{3}}$

Ans. *(b)* $\frac{1}{3(2)^{2/3}}$

Explanation: We have $\lim_{x \to 2} \frac{(x+2)^{1/3}-2^{1/3}}{(x+2)-2}$

So, we can write $\lim_{x \to 2} \frac{(x+2)^{1/3}-2^{1/3}}{(x+2)-2}$

Using $\lim_{x \to a} \frac{x^n-a^n}{x-a} = na^{n-1}$, we get

$$= \frac{1}{3}(2)^{\frac{1}{3}-1}$$

$$= \frac{1}{3}2^{\frac{-2}{3}}$$

$$= \frac{1}{3.(2)^{2/3}}$$

8. The value of $\lim_{x \to 0}\left(\frac{\tan^2 3x}{x^2}\right)$ is:

(a) 0 (b) 3
(c) ∞ (d) 9 [Diksha]

Ans. *(d) 9*

Explanation: Given, $\lim_{x \to 0}\left(\frac{\tan^2 3x}{x^2}\right)$

We know that, $\lim_{x \to 0} \frac{\tan ax}{ax} = 1$

$$= \lim_{x \to 0} 9\frac{\tan^2 3x}{9x^2}$$

$$= \lim_{x \to 0} 9\left(\frac{\tan 3x}{3x}\right)^2$$

$$= 9 \times 1$$

$$= 9$$

9. The value of $\lim_{x \to 0}\left[\frac{\sqrt{3+x}-\sqrt{3}}{x}\right]$ is:

(a) $\frac{1}{3(2)^{\frac{1}{2}}}$ (b) $\frac{1}{2(3)^{\frac{1}{2}}}$

(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

[Delhi Gov. QB 2022]

Ans. *(b)* $\frac{1}{2(3)^{\frac{1}{2}}}$

Explanation: We have, $\lim_{x \to 0}\left[\frac{\sqrt{3+x}-\sqrt{3}}{x}\right]$

We can write as $\frac{(3+x)^{1/2}-(3)^{1/2}}{(x+3)-(3)}$

Now, using $\lim_{x \to a}\frac{x^n-a^n}{x-a}=na^{n-1}$

Thus, $\frac{1}{2}(3)^{1/2-1}=\frac{1}{2}(3)^{\frac{-1}{2}}$

$$=\frac{1}{2(3)^{\frac{1}{2}}}$$

10. $\lim_{x \to 0}\frac{1-\cos 4x}{x^2}$ is:

(a) 8 (b) 1

(c) 2 (d) 3

[Delhi Gov. Term-1 SQP 2021]

Ans. *(a) 8*

Explanation: We have, $\frac{1-\cos 4x}{x^2}=\frac{2\sin^2 2x}{x^2}$

$$=2\left(\frac{\sin 2x}{2x}\right)^2\times 4$$

$$=8(1)^2=8$$

11. The value of $\lim_{x \to 1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$ is:

(a) $\frac{1}{10}$ (b) $\frac{-1}{10}$

(c) 1 (d) None of these

[NCERT Exemplar]

Ans. *(b)* $\frac{-1}{10}$

Explanation: Given, $\lim_{x \to 1}\frac{(\sqrt{x}-1)(2x-3)}{2x^2+x-3}$

$$=\lim_{x \to 1}\frac{(\sqrt{x}-1)(2x-3)}{x(2x+3)-1(2x+3)}$$

$$=\lim_{x \to 1}\frac{(\sqrt{x}-1)(2x-3)}{(x-1)(2x+3)}$$

$$=\lim_{x \to 1}\frac{(\sqrt{x}-1)(\sqrt{x}+1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)}$$

$$=\lim_{x \to 1}\frac{(x-1)(2x-3)}{(x-1)(\sqrt{x}+1)(2x+3)}$$

$$=\lim_{x \to 1}\frac{2x-3}{(\sqrt{x}+1)(2x+3)}$$

Taking limit we have

$$=\frac{2(1)-3}{(\sqrt{1}+1)(2\times 1+3)}=\frac{-1}{2\times 5}=\frac{-1}{10}$$

12. $\lim_{x \to 0}\frac{2\sin^2 3x}{x^2}$ is equal to:

(a) 12 (b) 18

(c) 0 (d) 6

[Delhi Gov. Term-1 SQP 2022]

Ans. *(b) 18*

Explanation:

$$\lim_{x \to 0}\frac{2\sin^2 3x}{x^2}=\lim_{x \to 0}2\left(\frac{\sin 3x}{x}\right)^2$$

$$=\lim_{x \to 0}2\times 9\left(\frac{\sin 3x}{3x}\right)^2$$

$$=18\left(\lim_{x \to 0}\frac{\sin 3x}{3x}\right)^2=18$$

13. If $f(x)=\begin{cases}x^2-1, & 0<x<2\\ 2x+3, & 2\le x\le 3\end{cases}$, then the quadratic equations, whose roots are $\lim_{x \to 2^-}f(x)$ and $\lim_{x \to 2^+}f(x)$ is:

(a) $x^2-6x+9=0$ (b) $x^2-7x+8=0$

(c) $x^2-14x+49=0$ (d) $x^2-10x+21=0$

[NCERT Exemplar]

Ans. *(d)* $x^2-10x+21=0$

Explanation: Given, $\begin{cases}x^2-1, & 0<x<2\\ 2x+3, & 2\le x\le 3\end{cases}$

$$\lim_{x \to 2^-}f(x)=\lim_{x \to 2^-}(x^2-1)$$

$$\lim_{h \to 0}[(2-h)^2-1]=\lim_{h \to 0}(4+h^2-4h-1)$$

$$=\lim_{h \to 0}(h^2-4h+3)=3$$

and $\lim_{x \to 2^+}f(x)=\lim_{x \to 2^+}(2x+3)$

$$=\lim_{h \to 0}[2(2+h)+3]$$

$$=7$$

Therefore, the quadratic equation whose roots are

3 and 7 is $x^2-(3+7)x+3\times 7=0$

i.e., $x^2-10x+21=0.$

14. $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x}$ is:

(a) 2 (b) $\frac{1}{2}$

(c) $-\frac{1}{2}$ (d) $\frac{1}{4}$

[NCERT Exemplar]

Ans. *(b)* $\frac{1}{2}$

Explanation: Given, $\lim_{x \to 0} \frac{\tan 2x - x}{3x - \sin x} = \left(\frac{0}{0}\right)$ form.

Now, applying *L* 'Hospital's rule we get,

$$= \lim_{x \to 0} \frac{2\sec^2 2x - 1}{3 - \cos x}$$

$$= \frac{2-1}{2}$$

$$= \frac{1}{2}$$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of thefollowing choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of (A).

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

15. Assertion (A): $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$ is equal to 1, where $a + b + c \neq 0$.

Reason (R): $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$ is equal to $\frac{1}{4}$

Ans. *(c) (A) is true but (R) is false.*

Explanation: Given, $\lim_{x \to 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$

$$= \frac{a \times (1)^2 + b \times 1 + c}{c \times (1)^2 + b \times 1 + a}$$

$$= \frac{a + b + c}{c + b + a} = 1$$

Given, $\lim_{x \to -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

$$= \lim_{x \to -2} \frac{(2 + x)}{2x(x + 2)} = \lim_{x \to -2} \frac{1}{2x}$$

$$= \frac{1}{2(-2)} = -\frac{1}{4}$$

16. Assertion (A): $\lim_{x \to 0} \frac{\sin ax}{bx}$ is equal to $\frac{a}{b}$.

Reason (R): $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: Given, $\lim_{x \to 0} \frac{\sin ax}{bx} = \lim_{x \to 0} \frac{(a)\sin ax}{b(ax)}$

[dividing and multiplying by *a*]

$$= \frac{a}{b} \times 1 = \frac{a}{b} \left[\because \lim_{x \to 0} \frac{\sin ax}{ax} = 1\right]$$

17. Assertion (A): $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$ is equal to –2.

Reason (R): $\lim_{x \to 1} (5x^3 + 5x + 1)$ is equal to 11.

Ans. *(d) (A) is false but (R) is true.*

Explanation: $\lim_{x \to 0} \frac{\sin ax + bx}{ax + \sin bx}$

Dividing each term by *x*, we get

$$= \lim_{x \to 0} \frac{\frac{\sin ax}{x} + \frac{bx}{x}}{\frac{ax}{x} + \frac{\sin bx}{x}} = \lim_{x \to 0} \frac{\frac{a \sin ax}{ax} + b}{a + \frac{b \sin bx}{bx}}$$

$$= \frac{a \times 1 + b}{a + b \times 1} = \frac{a + b}{a + b} = 1 \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

Given, $\lim_{x \to 1} (5x^3 + 5x + 1)$

$$= 5(1)^3 + 5(1) + 1 = 5 + 5 + 1 = 11$$

18. Assertion (A): $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$ is equal to π.

Reason (R): $\lim_{x \to 0} \frac{\cos x}{\pi - x}$ is equal to $\frac{1}{\pi}$.

Ans. *(d) (A) is false but (R) is true.*

Explanation: Given, $\lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

Let, $\pi - x = h$, As $x \to \pi$, then $h \to 0$.

$$\therefore \lim_{x \to \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{h \to 0} \frac{\sin h}{\pi h}$$

$$= \lim_{h \to 0} \frac{1}{\pi} \times \frac{\sin h}{h}$$

$$= \frac{1}{\pi} \times 1 = \frac{1}{\pi} \quad \left[\because \lim_{h \to 0} \frac{\sin h}{h} = 1\right]$$

Given, $\lim_{x \to 0} \frac{\cos x}{\pi - x}$

Putting, the limit directly, we get

$$\frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

CASE BASED Questions (CBQs)

[**4** & **5** marks]

Read the following passages and answer the questions that follow:

19. *A function f is said to be a rational function, if*

$f(x) = \frac{g(x)}{h(x)}$, where g(x) and h(x) are polynomial functions such that $h(x) \neq 0$.

Then, $\lim_{x \to a} f(x) = \lim_{x \to a} \frac{g(x)}{h(x)}$

$$= \frac{\lim_{x \to a} g(x)}{\lim_{x \to a} h(x)} = \frac{g(a)}{h(a)}$$

However, if $h(a) = 0$, then there are two cases arise,

(i) $g(a) \neq 0$ (ii) $g(a) = 0$

In the first case, we say that the limit does not exist.

In the second case, we can find limit.

(A) Find the value of $\lim_{x \to -2} \left[\frac{x^2 - 4}{x^3 - 4x^2 + 4x}\right]$.

(B) Find $\lim_{x \to -1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$.

(C) Find the positive integer "n" so that $\lim_{x \to 3} \left[\frac{(x^n - 3^n)}{(x - 3)}\right] = 108$.

Ans. (A) Consider $f(x) = \frac{x^2 - 4}{x^3 - 4x^2 + 4x}$

On putting $x = 2$, we get

$f(2) = \frac{4 - 4}{8 - 16 + 8} = \frac{0}{0}$ i.e., it is of the form $\frac{0}{0}$.

So, let us first factorise it.

Consider, $\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 4x^2 + 4x}$

$$= \lim_{x \to 2} \frac{(x+2)(x-2)}{x(x-2)^2}$$

$$= \lim_{x \to 2} \frac{(x+2)}{x(x-2)}$$

$$= \frac{2+2}{2(2-2)} = \frac{4}{0}$$

which is not defined.

$\therefore \lim_{x \to 2} \left[\frac{x^2 - 4}{x^3 - 4x^2 + 4x}\right]$ does not exist.

(B) Given $\lim_{x \to 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} \quad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \to 1} \frac{x^7 - x^5 - x^5 + 1}{x^3 - x^2 - 2x^2 + 2}$$

$$= \lim_{x \to 1} \frac{x^5(x^2 - 1) - 1(x^5 - 1)}{x^2(x - 1) - 2(x^2 - 1)}$$

On dividing numerator and denominator by $(x - 1)$, we get

$$= \lim_{x \to 1} \frac{\frac{x^5(x^2 - 1)}{(x - 1)} - \frac{1(x^5 - 1)}{(x - 1)}}{\frac{x^2(x - 1)}{(x - 1)} - \frac{2(x^2 - 1)}{(x - 1)}}$$

$$= \frac{\lim_{x \to 1} x^5(x + 1) - \lim_{x \to 1}\left(\frac{x^5 - 1}{x - 1}\right)}{\lim_{x \to 1} x^2 - \lim_{x \to 1} 2(x + 1)}$$

$$= \frac{1\times2-5\times(1)^4}{1-2\times2} = \frac{2-5}{1-4}$$

$$\left[\because \lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$= \frac{-3}{-3} = 1$$

(C) Given limit: $\lim_{x\to3}\left[\frac{(x^n-3^n)}{(x-3)}\right] = 108$

Now, we have

$$\lim_{x\to3}\left[\frac{(x^n-3^n)}{(x-3)}\right] = n(3)^{n-1}$$

$$n(3)^{n-1} = 108$$

Now, this can be written as:

$$n(3)^{n-1} = 4(27) = 4(3)^{4-1}$$

Therefore, by comparing the exponents in the above equation, we get:

$$n = 4$$

Therefore, the value of positive integer "n" is 4.

20. *Raj was learning limit of a polynomial function from his tutor Rajesh.*

His tutor told that a function f is said to be a polynomial function, if f(x) is zero function.

Now, let

$f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial function, where a_i's are real numbers and $a_n \neq 0$.

Then, limit of a polynomial function f(x)

$$= \lim_{x\to a} f(x)$$

$$= \lim_{x\to a} [a_0 + a_1x + a_2x^2 + ... + a_nx^n]$$

$$= \lim_{x\to a} a_0 + \lim_{x\to a} a_1x + \lim_{x\to a} a_2x^2 + ... + \lim_{x\to a} a_nx^n$$

$$= a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + ... + a_n \lim_{x\to a} x^n$$

$$= a_0 + a_1a + a_2a^2 + ... + a_na^n = f(a)$$

Based on above information, answer the following questions.

(A) $\lim_{x\to-1} (1 + x + x^2 + ... + x^9)$ is equal to:

(a) 0	(b) 1
(c) 2	(d) 3

(B) $\lim_{x\to5} [x^2(x-1)]$ is equal to:

(a) 10	(b) 100
(c) 25	(d) 125

(C) $\lim_{x\to2} (x^3 + x^2 + x - 1)$ is equal to:

(a) 9	(b) 11
(c) 10	(d) 13

(D) $\lim_{x\to-3} (x^3 + x + 2)$ is equal to:

(a) 28	(b) −28
(c) 30	(d) −15

(E) $\lim_{x\to4} (x^4 - x^3)$ is equal to:

(a) 192	(b) 180
(c) 50	(d) 165

Ans. (A) *(a)* 0

Explanation: Given, limit

$$= \lim_{x\to-1} (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9)$$

$$= 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 = 0$$

(B) *(b)* *100*

Explanation: Given, limit

$$= \lim_{x\to5} [x^2(x-1)]$$

$$= \lim_{x\to5} [x^3 - x^2]$$

$$= \lim_{x\to5} x^3 - \lim_{x\to5} x^2$$

$$= (5)^3 - (5)^2$$

$$= 125 - 25 = 100$$

(C) *(d)* *13*

Explanation: Given, limit

$$= \lim_{x\to2} (x^3 + x^2 + x - 1)$$

$$= \lim_{x\to2} x^3 + \lim_{x\to2} x^2 + \lim_{x\to2} x + \lim_{x\to2} (-1)$$

$$= (2)^3 + (2)^2 + (2) - 1$$

$$= 8 + 4 + 2 - 1 = 13$$

(D) *(b)* *−28*

Explanation: Given, limit

$$= \lim_{x\to-3} (x^3 + x + 2)$$

$$= \lim_{x\to-3} x^3 + \lim_{x\to-3} x + \lim_{x\to-3} 2$$

$= (-3)^3 + (-3) + 2$

$= -27 - 3 + 2$

$= -30 + 2 = -28$

(E) (a) 192

Explanation: Given, limit

$= \lim_{x \to 4} (x^4 - x^3)$

$= \lim_{x \to 4} x^4 - \lim_{x \to 4} x^3 = (4)^4 - (4)^3$

$= 256 - 64 = 192$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

21. Evaluate: $\lim_{x \to 0} \left[\frac{9x \cos x - 3\sin x}{4x^2 - \tan x}\right]$.

Ans. Given,

$$\lim_{x \to 0} \left[\frac{9x \cos x - 3\sin x}{4x^2 - \tan x}\right] = \lim_{x \to 0} \left[\frac{9 \cos x - 3\dfrac{\sin x}{x}}{4x - \dfrac{\tan x}{x}}\right]$$

[On dividing the numerator and denominator by x]

$$= \frac{9 \lim_{x \to 0} [\cos x] - 3 \lim_{x \to 0} \left[\frac{\sin x}{x}\right]}{4 \lim_{x \to 0} [x] - \lim_{x \to 0} \left[\frac{\tan x}{x}\right]} = \frac{9(1) - 3(1)}{4(0) - (1)} = -6$$

22. Evaluate: $\lim_{x \to \pi/6} \left[\frac{\cos^2 x - 4}{\cos x - 2}\right]$.

Ans. Given,

$$\lim_{x \to \pi/6} \left[\frac{\cos^2 x - 4}{\cos x - 2}\right] = \lim_{x \to \pi/6} \left[\frac{(\cos x + 2)(\cos x - 2)}{\cos x - 2}\right]$$

$$= \lim_{x \to \pi/6} (\cos x + 2)$$

$$= \lim_{x \to \pi/6} [\cos x + 2] = \cos\frac{\pi}{6} + 2$$

$$= \frac{\sqrt{3}}{2} + 2$$

23. Evaluate: $\lim_{x \to 0} \left(\frac{\sqrt{2+x} - \sqrt{2}}{x}\right)$. **[Diksha]**

Ans. Given, $\lim_{x \to 0} \left(\frac{\sqrt{2+x} - \sqrt{2}}{x}\right)$

$$= \lim_{x \to 0} \left(\frac{\sqrt{2+x} - \sqrt{2}}{x}\right) \times \left(\frac{\sqrt{2+x} + \sqrt{2}}{\sqrt{2+x} + \sqrt{2}}\right)$$

$$= \lim_{x \to 0} \left(\frac{(2+x) - 2}{x(\sqrt{2+x} + \sqrt{2})}\right)$$

$$= \lim_{x \to 0} \left(\frac{1}{\sqrt{2+x} + \sqrt{2}}\right)$$

$$= \left(\frac{1}{2\sqrt{2}}\right)$$

24. Evaluate: $\lim_{x \to 8} \left[\frac{e^x - e^4}{x - 4}\right]$.

Ans. Let $L = \lim_{x \to 8} \left[\frac{e^x - e^4}{x - 4}\right]$

Putting, $x = 4 + y$

Then, $y = x - 4$ and $y \to 0$ and as $x \to 4$

$$\therefore \quad L = \lim_{y \to 0} \left[\frac{e^{4+y} - e^4}{y}\right] = \lim_{y \to 0} \left[\frac{e^4 e^y - e^4}{y}\right]$$

$$= \lim_{y \to 0} \left[\frac{e^4 (e^y - 1)}{y}\right]$$

$$= e^4 \lim_{y \to 0} \left[\frac{e^y - 1}{y}\right] = e^4(1) = e^4.$$

25. Evaluate: $\lim_{x \to 0} \left[\frac{\log(1+5x)}{3x}\right]$.

Ans. $\lim_{x \to 0} \left[\frac{\log(1+5x)}{3x}\right] = \lim_{x \to 0} \left[\frac{5}{3} \times \frac{\log(1+5x)}{5x}\right]$

{Multiplying and dividing by 5}

$$= \frac{5}{3} \lim_{x \to 0} \left[\frac{\log(1+5x)}{5x}\right]$$

$$= \frac{5}{3} \lim_{5x \to 0} \left[\frac{\log(1+5x)}{5x}\right]$$

[$\because$ $5x \to 0$ as $x \to 0$]

$$= \frac{5}{3}(1) = \frac{5}{3}$$

26. Evaluate: $\lim_{x \to a} \frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{x-a}$.

[NCERT Exemplar]

Ans. $\lim_{x \to a} \frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{(2+x)-(a+2)}$

$$= \lim_{2+x \to a+2} \frac{(2+x)^{\frac{5}{2}}-(a+2)^{\frac{5}{2}}}{(2+x)-(a+2)}$$

$$= \frac{5}{2}(a+2)^{\frac{5}{2}-1}$$

$$= \frac{5}{2}(a+2)^{\frac{3}{2}}$$

$$\left[\because \lim_{x \to a} \frac{x^n-a^n}{x-a} = n \cdot a^{n-1}\right]$$

27. Evaluate: $\lim_{x \to 0} \frac{\sin(2x)+3x}{2x+(\tan 3x)}$.

[Delhi Gov. QB 2022]

Ans. Given,

$$\lim_{x \to 0} \frac{\sin(2x)+3x}{2x+(\tan 3x)} = \lim_{x \to 0} \frac{\frac{2x(\sin 2x)}{2x}+3x}{2x+\frac{3x(\tan 3x)}{3x}}$$

$$= \frac{\lim_{x\to 0} 2x . \lim_{x \to 0}\left[\frac{\sin 2x}{2x}\right] + \lim_{x\to 0} 3x}{\lim_{x\to 0} 2x + \lim_{x\to 0} 3x . \lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]}$$

Now as $\lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]$ and $\lim_{x\to 0}\left[\frac{\sin 2x}{2x}\right]$ both will be 1.

$$\Rightarrow \frac{\lim_{x\to 0} 2x . \lim_{x \to 0}\left[\frac{\sin 2x}{2x}\right] + \lim_{x\to 0} 3x}{\lim_{x\to 0} 2x + \lim_{x\to 0} 3x . \lim_{x\to 0}\left[\frac{\tan 3x}{3x}\right]}$$

$$= \frac{\lim_{x\to 0} 2x . 1 + \lim_{x\to 0} 3x}{\lim_{x\to 0} 2x + \lim_{x\to 0} 3x . 1}$$

$$= \lim_{x\to 0} \frac{2x+3x}{2x+3x} = \lim_{x\to 0} 1 = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin 2x + 3x}{2x+\tan 3x} = 1$$

28. Evaluate: $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$. [NCERT Exemplar]

Ans. Given that, $\lim_{x \to \frac{\pi}{3}} \frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sqrt{2\sin^2 3x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)} \quad \left[\because 1-\cos\theta = 2\sin^2\frac{\theta}{2}\right]$$

$$= \lim_{x \to \frac{\pi}{3}} \frac{\sqrt{2}\sin 3x}{\sqrt{2}\left(\frac{\pi-3x}{3}\right)} = \lim_{x \to \frac{\pi}{3}} \frac{3 \cdot \sin(\pi-3x)}{\pi-3x}$$

$$= 3 \times 1 = 3 \quad \left[\because \lim_{x\to 0}\frac{\sin x}{x} = 1\right]$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

29. If $\lim_{x \to 1}\left[\frac{x^4-1}{x-1}\right] = \lim_{x\to k}\left[\frac{x^3-k^3}{x^2-k^2}\right]$, then find the value of k. [NCERT Exemplar]

Ans. Given that, $\lim_{x \to 1}\left[\frac{x^4-1}{x-1}\right] = \lim_{x\to k}\left[\frac{x^3-k^3}{x^2-k^2}\right]$

$$\lim_{x \to 1}\frac{x^4-1^4}{x-1} = \lim\left[\frac{x^3-k^3}{x^2-k^2}\right]$$

$$\left[\because \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$\Rightarrow \quad 4(1)^{4-1} = \lim_{x\to k}\left[\frac{\left(\frac{x^3-k^3}{x-k}\right)}{\left(\frac{x^2-k^2}{x-k}\right)}\right]$$

$$\Rightarrow \quad 4 = \frac{3k^2}{2k}$$

$$\left[\because \frac{x^n-a^n}{x-a} = na^{n-1}\right]$$

$$\Rightarrow \quad 4 = \frac{3k}{2}$$

$$\Rightarrow \quad k = \frac{8}{3}.$$

30. Evaluate the given limit: $\lim_{x\to 3}\dfrac{x^2-9}{2x^2-5x-3}$.

Ans. Given, $\lim_{x\to 3}\dfrac{x^2-9}{2x^2-5x-3}$

$$\lim_{x\to 3}\frac{x^2-9}{2x^2-5x-3}=\lim_{x\to 3}\frac{(x)^2-(3)^2}{2x^2-6x+x-3}$$

$$=\lim_{x\to 3}\frac{(x-3)(x+3)}{2x(x-3)+1(x-3)}$$

$$=\lim_{x\to 3}\frac{(x+3)}{(2x+1)}$$

On putting $x = 3$, we get

$$=\frac{(3+3)}{2\times 3+1}$$

$$=\frac{6}{6+1}$$

$$=\frac{6}{7}$$

31. Find $\lim_{x\to 0}[f(x)]$, where

$$f(x)=\begin{cases}2x+3 \text{ if } x\le 0\\ 3(x+1) \text{ if } x>0\end{cases}$$

Ans. Given

R.H.L. $=\lim_{x\to 0+}[f(x)]$

$=\lim_{x\to 0+}[3(x+1)]$

$=\lim_{x\to 0}[3(0+h+1)]$ [By putting $x = 0 + h$]

L.H.L. $=\lim_{x\to 0-}[f(x)]$

$=\lim_{x\to 0-}[2x+3]$

$=\lim_{h\to 0}[2(0-h)+3]$ [By putting $x = 0 - h$]

$= 3$

Since, $\lim_{x\to 0+}[f(x)]=\lim_{x\to 0-}[f(x)]=3$

Hence, $\lim_{x\to 0}[f(x)]=3$

32. Find $\lim_{x\to 5}[f(x)]$, where $f(x) = |x| - 5$.

Ans. Given $f(x)=|x|-5=\begin{cases}x-5 & \text{if } x\ge 0\\ -x-5 & \text{if } x<0\end{cases}$

R.H.L. $=\lim_{x\to 5+}[f(x)]$

$=\lim_{x\to 5+}[x-5]$

$=\lim_{x\to 0}[5+h-5]$ [But putting $x = 5 + h$]

L.H.L. $=\lim_{x\to 5-}[f(x)]$

$=\lim_{x\to 5-}[x-5]$

$=\lim_{h\to 0}[5-h-5]$ [But putting $x = 5 - h$]

$= 0$

Since, $\lim_{x\to 5+}[f(x)]=\lim_{x\to 5-}[f(x)]=0$

Hence, $\lim_{x\to 5}[f(x)=0]$

33. Evaluate: $\lim_{x\to 0}\dfrac{\sin 2x+3x}{2x+\tan 3x}$. **[NCERT Exemplar]**

Ans. Given that: $\lim_{x\to 0}\dfrac{\sin 2x+3x}{2x+\tan 3x}$

$$=\lim_{x\to 0}\frac{\left(\dfrac{\sin 2x+3x}{2x}\right)\times 2x}{\left(\dfrac{2x+\tan 3x}{3x}\right)\times 3x}$$

$$=\lim_{x\to 0}\frac{\left(\dfrac{\sin 2x}{2x}+\dfrac{3x}{2x}\right)\times 2x}{\left(\dfrac{2x}{3x}+\dfrac{\tan 3x}{3x}\right)\times 3x}$$

$$=\frac{\left(\lim_{x\to 0}\dfrac{\sin 2x}{2x}+\dfrac{3}{2}\right)}{\left[\dfrac{2}{3}+\lim_{x\to 0}\dfrac{\tan 3x}{3x}\right)}\times\frac{2}{3}$$

$$=\frac{\left(1+\dfrac{3}{2}\right)}{\left(\dfrac{2}{3}+1\right)}\times\frac{2}{3}$$

$$=\frac{\frac{5}{2}}{\frac{5}{3}}\times\frac{2}{3}=\frac{3}{2}\times\frac{2}{3}=1 \qquad \left[\because \lim_{x\to 0}\frac{\sin x}{x}=1\right]$$

34. Evaluate: $\lim_{x\to\frac{1}{2}}\left(\dfrac{8x-3}{2x-1}-\dfrac{4x^2+1}{4x^2-1}\right)$.

[NCERT Exemplar]

Ans. Given that, $\lim_{x\to\frac{1}{2}}\left(\dfrac{8x-3}{2x-1}-\dfrac{4x^2+1}{4x^2-1}\right)$

$$=\lim_{x\to\frac{1}{2}}\left[\frac{(8x-3)(2x+1)-(4x^2+1)}{(4x^2-1)}\right]$$

$$= \lim_{x \to \frac{1}{2}} \left[\frac{16x^2 - 6x + 8x - 3 - 4x^2 - 1}{4x^2 - 1}\right]$$

$$= \lim_{x \to \frac{1}{2}} \left[\frac{12x^2 + 2x - 4}{4x^2 - 1}\right] = \lim_{x \to \frac{1}{2}} \frac{2(6x^2 + x - 2)}{4x^2 - 1}$$

$$= \lim_{x \to \frac{1}{2}} \frac{2\left[6x^2 + 4x - 3x - 2\right]}{(2x+1)(2x-1)}$$

$$= \lim_{x \to \frac{1}{2}} \frac{2\left[2x(3x+2) - 1(3x+2)\right]}{(2x+1)(2x-1)}$$

$$= \lim_{x \to \frac{1}{2}} \frac{2(3x+2)(2x-1)}{(2x+1)(2x-1)} = \lim_{x \to \frac{1}{2}} \frac{2(3x+2)}{(2x+1)}$$

Taking limit, we have

$$= \frac{2\left(3 \times \frac{1}{2} + 2\right)}{2 \times \frac{1}{2} + 1} = \frac{2\left(\frac{7}{2}\right)}{2} = \frac{7}{2}$$

SHORT ANSWER Type-II Questions (SA-II)

[**3** marks]

35. Evaluate: $\lim_{x \to 0} \left[\frac{\sin 5x - \sin 3x}{\tan 7x}\right]$.

Ans. Given,

$$\lim_{x \to 0} \left[\frac{\sin 5x - \sin 3x}{\tan 7x}\right] = \lim_{x \to 0} \left[\frac{\frac{\sin 5x}{x} - \frac{\sin 3x}{x}}{\frac{\tan 7x}{x}}\right]$$

[On dividing numerator and denominator by x]

$$= \lim_{x \to 0} \left[\frac{5\frac{\sin 5x}{5x} - 3\frac{\sin 3x}{3x}}{7\frac{\tan 7x}{7x}}\right]$$

$$= \frac{5 \lim_{x \to 0}\left[\frac{\sin 5x}{5x}\right] - 3 \lim_{x \to 0}\left[\frac{\sin 3x}{3x}\right]}{7 \lim_{x \to 0}\left[\frac{\tan 7x}{7x}\right]}$$

$$= \frac{5 \lim_{5x \to 0}\left[\frac{\sin 5x}{5x}\right] - 3 \lim_{3x \to 0}\left[\frac{\sin 3x}{3x}\right]}{7 \lim_{7x \to 0}\left[\frac{\tan 7x}{7x}\right]}$$

[$\because 5x \to 0,\ 3x \to 0,\ 7x \to 0$ as $x \to 0$]

$$= \frac{5(1) - 3(1)}{7(1)} = \frac{2}{7}$$

36. Find $\lim_{x \to 0} [f(x)]$, **where** $f(x) = \begin{cases} \frac{|x|}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$.

Ans. $f(x) = \begin{cases} \frac{|x|}{x} \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases} = \begin{cases} \frac{x}{x} \text{ if } x > 0 \\ -\frac{x}{x} \text{ if } x = 0 \\ 0 \text{ if } x = 0 \end{cases} = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ 0 \text{ if } x = 0 \end{cases}$

Given, R.H.L. $= \lim_{x \to 0} [f(x)]$

$$= \lim_{x \to 0} [1] = 1$$

L.H.L. $= \lim_{x \to 0} [f(x)]$

$$= \lim_{x \to 0} [-1] = -1$$

Since, $\lim_{x \to 0} [f(x)] \neq \lim_{x \to 0} [f(x)]$

Hence, $\lim_{x \to 0} [f(x)]$ does not exist.

37. Evaluate: $\lim_{x \to 0} \left[\frac{\tan x - 2\tan 3x + \tan 5x}{x}\right]$.

Ans. Given, $\lim_{x \to 0} \left[\frac{\tan x - 2\tan 3x + \tan 5x}{x}\right]$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x} - \frac{2\tan 3x}{x} + \frac{\tan 5x}{x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x} - 6\frac{\tan 3x}{3x} + 5\frac{\tan 5x}{5x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x}\right] - 6 \lim_{x \to 0} \left[\frac{\tan 3x}{3x}\right] + 5 \lim_{x \to 0} \left[\frac{\tan 5x}{5x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\tan x}{x}\right] - 6 \lim_{3x \to 0} \left[\frac{\tan 3x}{3x}\right] + 5 \lim_{5x \to 0} \left[\frac{\tan 5x}{5x}\right]$$

$= 1 - 6(1) + 5(1) = 0$

38. Evaluate: $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$. **[NCERT Exemplar]**

Ans. Given, $\lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}}$

$$= \lim_{x \to a} \frac{\sin x - \sin a}{\sqrt{x} - \sqrt{a}} \times \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}}$$

$$= \lim_{x \to a} \frac{(\sin x - \sin a)(\sqrt{x}+\sqrt{a})}{x-a}$$

$$= \lim_{x \to a} \frac{\left(2\cos\frac{x+a}{2}\cdot\sin\frac{x-a}{2}\right)(\sqrt{x}+\sqrt{a})}{x-a}$$

$$= \lim_{x \to a} \left(2\cos\frac{x+a}{2}\cdot\frac{\sin\frac{x-a}{2}}{2\times\frac{x-a}{2}}\right)(\sqrt{x}+\sqrt{a})$$

$$= \lim_{x \to a} \cos\left(\frac{x+a}{2}\right)(\sqrt{x}+\sqrt{a})$$

$$\left[\because \lim_{\frac{x-a}{2} \to 0} \frac{\sin\frac{x-a}{2}}{\frac{x-a}{2}} = 1\right]$$

Taking limit we have

$$= \cos\left(\frac{a+a}{2}\right)(\sqrt{a}+\sqrt{a})$$

$$= \cos a \times 2\sqrt{a}$$

$$= 2\sqrt{a}\cdot\cos a$$

39. Evaluate: $\lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$.

[NCERT Exemplar]

Ans. Given, that $\lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x}$

$$= \lim_{x \to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{\sin^2 x} \times \frac{\sqrt{2}+\sqrt{1+\cos x}}{\sqrt{2}+\sqrt{1+\cos x}}$$

$$= \lim_{x \to 0} \frac{2-(1+\cos x)}{\sin^2 x\left[\sqrt{2}+\sqrt{1+\cos x}\right]}$$

$$= \lim_{x \to 0} \frac{1-\cos x}{\sin^2 x\left[\sqrt{2}+\sqrt{1+\cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x/2}{(2\sin x/2\cos x/2)^2} \times \frac{1}{\left[\sqrt{2}+\sqrt{1+\cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2\sin^2 x/2}{(4\sin^2 x/2\cos^2 x/2)^2} \times \frac{1}{\left[\sqrt{2}+\sqrt{1+\cos x}\right]}$$

$$= \lim_{x \to 0} \frac{2}{4\cos^2\frac{x}{2}} \times \frac{1}{\left[\sqrt{2}+\sqrt{1+\cos x}\right]}$$

Taking limit, we get

$$= \frac{2}{4\cos^2 0} \times \frac{1}{(\sqrt{2}+\sqrt{2})} = \frac{1}{2} \times \frac{1}{2\sqrt{2}} = \frac{1}{4\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}}$$

LONG ANSWER Type Questions (LA)

[**4** & **5** marks]

40. Evaluate: $\lim_{x \to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$.

Ans. When, $x=3$, the expression $\lim_{x \to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27}$

assumes the form $\frac{0}{0}$.

So, $(x - 3)$ is a factor of numerator and denominator. Factorising the numerator and denominator, we get

$$\lim_{x \to 3} \frac{x^3-7x^2+15x-9}{x^4-5x^3+27x-27} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x^2-4x+3)}{(x-3)(x^3-2x^2-6x+9)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x^2-4x+3}{x^3-2x^2-6x+9} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{(x-3)(x-1)}{(x-3)(x^2+x-3)} \quad \left(\text{form } \frac{0}{0}\right)$$

$$= \lim_{x \to 3} \frac{x-1}{x^2+x-3} = \frac{3-1}{9+3-3} = \frac{2}{9}$$

41. Evaluate: $\lim_{x \to a} \left[\frac{\log x - \log a}{x-a}\right]$.

Ans. We have, $\lim_{x \to a} \left[\frac{\log x - \log a}{x-a}\right]$

Put $x = a + y$

Then, $y = x - a$ and $y \to 0$ as $x \to a$

$$\lim_{x\to 0}\left[\frac{\log x - \log a}{x-a}\right] = \lim_{y\to 0}\left[\frac{\log(a+y)-\log a}{y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\log\left(\frac{a+y}{a}\right)}{y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\log\left(1+\frac{y}{a}\right)}{a\left(\frac{y}{a}\right)}\right]$$

$$= \frac{1}{a}\lim_{y\to 0}\left[\frac{\log\left(1+\frac{y}{a}\right)}{\left(\frac{y}{a}\right)}\right]$$

$$= \frac{1}{a}\lim_{(y/a)\to 0}\left[\frac{\log\left(1+\frac{y}{a}\right)}{\left(\frac{y}{a}\right)}\right]$$

$$= \left[\because \frac{y}{a}\to 0 \text{ as } y\to 0\right]$$

$$= \frac{1}{a}(1) = \frac{1}{a}$$

42. **Evaluate:** $\lim_{x\to\pi/3}\left[\frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}\right]$.

Ans. Given:

$$\text{Let } L = \lim_{x\to\pi/3}\left[\frac{\sqrt{1-\cos 6x}}{\sqrt{2}\left(\frac{\pi}{3}-x\right)}\right]$$

On putting $x = \frac{\pi}{3} - y$, we get

Then, $y = \frac{\pi}{3} - x$ and $y \to 0$ as $x \to \frac{\pi}{3}$

$$\therefore \quad L = \lim_{y\to 0}\left[\frac{\sqrt{1-\cos(2\pi-6y)}}{\sqrt{2}y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\sqrt{1-\cos 6y}}{\sqrt{2}y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\sqrt{2\sin^2 3y}}{\sqrt{2}y}\right]$$

$$= \lim_{y\to 0}\left[\frac{\sin 3y}{y}\right]$$

$$= \lim_{y\to 0}\left[\frac{3\sin 3y}{3y}\right]$$

$$= 3 \times \lim_{y\to 0}\left[\frac{\sin 3y}{3y}\right]$$

$$= 3\times \lim_{3y\to 0}\left[\frac{\sin 3y}{3y}\right] \quad [\because 3y\to 0 \text{ as } y\to 0]$$

$$= 3(1) = 3$$

43. **Differentiate $\cos(x^2 + 1)$s with respect to 'x' using first principal method.**

[NCERT Exemplar]

Ans. Let $f(x) = \cos(x^2 + 1)$ ...(i)

$\Rightarrow f(x + \Delta x) = \cos[(x + \Delta x)^2 + 1]$ ...(ii)

Subtracting equation (i) from equation (ii) we get

$f(x + \Delta x) - f(x) = \cos[(x + \Delta x)^2 + 1] - \cos(x^2 + 1)$

Dividing both sides by Δx we get

$$\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{[\cos[(x+\Delta x)^2+1]-\cos(x^2+1)]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\cos[(x+\Delta x)^2+1)]-\cos(x^2+1)}{\Delta x}$$

$$= f'(x) = \lim_{\Delta x\to 0}\frac{\cos[(x+\Delta x)^2+1]-\cos(x^2+1)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-2\sin\left[\frac{(x+\Delta x)^2+1+x^2+1}{2}\right]\times\sin\left[\frac{(x+\Delta x)^2+1-x^2-1}{2}\right]}{\Delta x}$$

$$\left[\because \cos C - \cos D = -2\sin\frac{C+D}{2}.\sin\frac{C-D}{2}\right]$$

$$= \lim_{\Delta x\to 0}\frac{-2\sin\left[\frac{x^2+\Delta x^2+2x.\Delta x+x^2+2}{2}\right]\times\sin\left[\frac{x^2+\Delta x^2+2x.\Delta x-x^2}{2}\right]}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{-2\sin\frac{[x^2+\frac{\Delta x^2}{2}+x.\Delta x-1]}{2}\sin\left[\Delta x\frac{(\Delta x+2x)}{2}\right]}{\Delta x}$$

$$= -2\sin\left[x^2 + \frac{\Delta x^2}{2} + x\Delta x + 1\right]$$

$$= \lim_{\Delta x \to 0} \frac{\sin\left[\Delta x \frac{(\Delta x + 2x)}{2}\right]}{\Delta x\left[\frac{\Delta x + 2x}{2}\right]} \times \left(\frac{\Delta x + 2x}{2}\right)$$

$$= \lim_{\Delta x \to 0} -2\sin\left[x^2 + \frac{\Delta x^2}{2} + x\Delta x + 1\right]$$

$$.\sin\frac{\left[\Delta x \frac{(\Delta x + 2x)}{2}\right]}{\Delta x\left[\frac{\Delta x + 2x}{2}\right]} \times \left[\frac{\Delta x + 2x}{2}\right]$$

$$\therefore \quad \Delta x\left[\frac{\Delta x + 2x}{2}\right] \to 0$$

Taking limit, we have

$= -2 \sin (x^2 + 1) \cdot 1 \cdot (x)$

$= -2x \sin (x^2 + 1)$

$$\left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

DERIVATIVES 2

TOPIC 1

DERIVATIVE AT A POINT

Let $f(x)$ be a real valued function defined on an open interval (a, b) and let $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$, if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely.}$$

This limit is called the derivative or differentiation of $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\left\{\frac{d}{dx} f(x)\right\} x = c$

.

That is, $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, provided that the limit exists.

Throughout this chapter it will be assumed that a given function $f(x)$ is differentiable at every point in its domain,

i.e. $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists for all c in its domain.

$$\therefore \quad f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

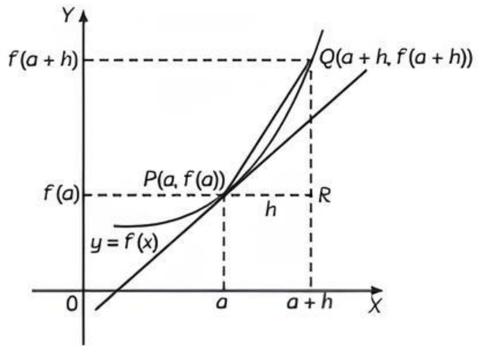
$$\Rightarrow \quad f'(c) = \lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$

$$\text{or,} \quad f'(c) = \lim_{h \to 0} \frac{f(c - h) - f(c)}{-h}$$

Geometrical Meaning of $\frac{dy}{dx}$

Let $P(a, f(a))$ be any point on the curve $y = f(x)$.

Let $Q(a + h, f(a + h))$ be any neighbouring point on the curve. Let the secant PQ be inclined at an angle θ with, x-axis, as shown in the figure.



Then, slope of secant $PQ = \tan\theta$

$$= \frac{QR}{PR}$$

$$= \frac{f(a + h) - f(a)}{h}.$$

As $h \to 0$, we observe that $Q \to P$ and secant $PQ \to$ tangent at point P.

So, slope of tangent at $P = \lim_{Q \to P}$ [slope of secant PQ]

$$\lim_{h \to 0}\left[\frac{f(a + h) - f(a)}{h}\right] = \left(\frac{dy}{dx}\right)_{x = a}$$

Conclusion: $\left(\frac{dy}{dx}\right)_{x = a}$ represent slope of tangent to the curve $y = f(x)$ at the point $(a, f(a))$.

In general, $\frac{dy}{dx}$ represents the slope of the tangent to the curve $y = f(x)$ at the point (x, y).

Basic Rules of Differentiation

In the previous section, we have discussed the method of finding the derivative using the first principle. In this section, we shall learn how to find the derivation without using the first principle directly.

Let us now summarise the results from the previous section.

(1) $\frac{d}{dx}(a) = 0$, where a is a constant.

(2) $\frac{d}{dx}(x^n) = nx^{n-1}$.

(3) $\frac{d}{dx}(\sin x) = \cos x$.

(4) $\frac{d}{dx}(\cos x) = -\sin x$.

(5) $\frac{d}{dx}(\tan x) = \sec^2 x$.

(6) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(7) $\frac{d}{dx}(\sec x) = \sec x . \tan x$.

(8) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x . \cot x$

Let us now state a very important theorem. This theorem tells us the method of finding the derivatives of sum, difference, product and quotient of functions.

Theorem

Let f and g be two real functions such that their derivatives are defined in domain D.

Let c be any real number. Then

(1) Derivative of the scalar multiple of a function is given by

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

(2) Derivative of the power of a function is given by

$$\frac{d}{dx}[\{f(x)\}^n] = n\{f(x)\}^{n-1}\frac{d}{dx}[f(x)].$$

(3) Derivative of sum of function is given by

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)].$$

(4) Derivative of difference of function is given by

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)].$$

(5) Derivation of product of function is given by

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

(6) Derivation of quotient of functions is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Whenever $g(x) \neq 0$.

Example 2.1: Find the derivative of the following functions w.r.t. x:

(A) $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

(B) $x^4 (5 \sin x - 3 \cos x)$

Ans. (A) Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$$\text{Then, } f'(x) = \frac{d}{dx}\left(\frac{a}{x^4} - \frac{b}{x^2} + \cos x\right)$$

$$= \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(\cos x)$$

$$= a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(\cos x)$$

$$= a(-4x^{-5}) - b.(-2x^{-3}) - \sin x$$

$$= -\frac{4a}{x^5} + \frac{2b}{x^3} - \sin x.$$

(B) Let $f(x) = x^4 (5 \sin x - 3 \cos x)$.

$$\text{Then, } f'(x) = \frac{d}{dx}[x^4 (5 \sin x - 3 \cos x)]$$

$$= x^4 \frac{d}{dx}(5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x)\frac{d}{dx}(x^4)$$

$$= x^4\left[\frac{d}{dx}(5 \sin x) - \frac{d}{dx}(3 \cos x)\right] + (5 \sin x - 3 \cos x)(4x^3)$$

$$= x^4 (5 \cos x + 3 \sin x) + (5 \sin x - 3 \cos x)(4x^3)$$

Example 2.2: Find the derivation of the following functions w.r.t x:

(A) $5 \sin x - 6 \cos x + 7$

(B) $2 \tan x - 7 \sec x$

Ans. (A) Let $f(x) = 5 \sin x - 6 \cos x + 7$

$$\text{Then, } f'(x) = \frac{d}{dx}(5 \sin x - 6 \cos x + 7)$$

$$= \frac{d}{dx}(5 \sin x) - \frac{d}{dx}(6 \cos x) + \frac{d}{dx}(7)$$

$$= 5\frac{d}{dx}(\sin x) - 6\frac{d}{dx}(\cos x) + 0$$

$$= 5 \cos x + 6 \sin x$$

(B) Let $f(x) = 2 \tan x - 7 \sec x$

$$\text{Then, } f'(x) = \frac{d}{dx}(2 \tan x - 7 \sec x)$$

$$= \frac{d}{dx}(2 \tan x) - \frac{d}{dx}(7 \sec x)$$

$$= 2\frac{d}{dx}(\tan x) - 7\frac{d}{dx}(\sec x)$$

$$= 2 \sec^2 x - 7 \sec x \tan x.$$

Example 2.3: Differentiate the following functions with respect to x:

(A) $\sin (x + a)$

(B) $\frac{\sin (x + a)}{\cos x}$

Ans. (A) Clearly, $\frac{d}{dx}[\sin(x+a)]$

$$= \frac{d}{dx}[\sin x \cos a + \cos x \sin a]$$

$$= \frac{d}{dx}(\sin x \cos a) + \frac{d}{dx}(\cos x \sin a)$$

$$= \cos a\frac{d}{dx}(\sin x) + \sin a\frac{d}{dx}(\cos x)$$

$$= \cos a \cos x + \sin a (-\sin x)$$

$$= \cos x \cos a - \sin x \sin a = \cos (x + a)$$

(B) Clearly, $\frac{d}{dx}\left\{\frac{\sin(x+a)}{\cos x}\right\}$

$$= \frac{d}{dx}\left\{\frac{\sin x \cos a + \cos x \sin a}{\cos x}\right\}$$

$$= \frac{d}{dx}\{\tan x \cos a + \sin a\}$$

$$= \frac{d}{dx}(\tan x \cos a) + \frac{d}{dx}(\sin a)$$

$$= \cos a \frac{d}{dx}(\tan x) + \frac{d}{dx}(\sin a)$$

$$= \cos a \times \sec^2 x + 0$$

$$= \sec^2 x \cos a$$

Example 2.4: Find the derivative of the $x^3 - 27$ function from first principles.

Ans. Let $f(x) = x^3 - 27$.

Then, $f(x+h) = (x+h)^3 - 27$

$$\therefore \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{\{(x+h)^3 - 27\} - (x^3 - 27)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x))$$

$$= \lim_{h \to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3 - 27) - (x^3 - 27)}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \lim_{h \to 0}(3x^2 + 3xh + h^2)$$

$$= 3x^2 + 3x \times 0 + 0 = 3x^2$$

Example 2.5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constant m and n are integers).

(A) $\dfrac{4x + 5\sin x}{3x + 7\cos x}$ (B) $(ax + b)^n$

(C) $\dfrac{px^2 + qx + r}{ax + b}$ (D) $\dfrac{a}{x^4} - \dfrac{b}{x^2} + \cos x$

Ans. (A) Let $f(x) = \dfrac{4x + 5\sin x}{3x + 7\cos x}$

By quotient rule,

$$f'(x) = \frac{(3x + 7\cos x)\frac{d}{dx}(4x + 5\sin x) - (4x + 5\sin x)\frac{d}{dx}(3x + 7\cos x)}{(3x + 7\cos x)^2}$$

$$= \frac{(3x + 7\cos x)\left[4\frac{d}{dx}(x) + 5\frac{d}{dx}(\sin x)\right] - (4x + 5\sin x)\left[3\frac{d}{dx}(x) + 7\frac{d}{dx}(\cos x)\right]}{(3x + 7\cos x)^2}$$

$$= \frac{(3x + 7\cos x)[4 + 5\cos x] - (4x + 5\sin x)[3 - 7\sin x]}{(3x + 7\cos x)^2}$$

$$= \frac{12x + 15x\cos x + 28\cos x + 35\cos^2 x - 12x + 28x\sin x - 15\sin x + 35\sin^2 x}{(3x + 7\cos x)^2}$$

$$= \frac{15x\cos x + 28\cos x + 28x\sin x - 15\sin x + 35(\cos^2 x + \sin^2 x)}{(3x + 7\cos x)^2}$$

$$= \frac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}$$

(B) Let $f(x) = (ax + b)^n$.

Accordingly, $f(x+h) = \{a(x+h) + b\}^n$

$$= (ax + ah + b)^n$$

By first principle,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(ax + ah + b)^n - (ax + b)^n}{h}$$

$$= \lim_{h \to 0} \frac{(ax + b)^n\left(1 + \frac{ah}{ax + b}\right)^n - (ax + b)^n}{h}$$

$$= (ax + b)^n \lim_{h \to 0} \frac{1}{h}\left[\left\{1 + \left(\frac{ah}{ax + b}\right) + \frac{n(n-1)}{2}\left(\frac{ab}{ax + b}\right)^2 + \ldots\right\} - 1\right]$$

(using binomial theorem)

$$= (ax + b)^n \lim_{h \to 0} \frac{1}{h}\left[n\left(\frac{ah}{ax + b}\right) + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \ldots \text{(terms containing higher degree of } h)\right]$$

$$= (ax + b)^n \lim_{h \to 0}\left[\frac{na}{(ax + b)} + \frac{n(n-a)a^2h^2}{2(ax + b)^2} + \ldots\right]$$

$$= (ax + b)^n\left[\frac{na}{(ax + b)} + 0\right]$$

$$= na\frac{(ax + b)^n}{ax + b}$$

$$= na(ax + b)^{(n-1)}$$

(C) Let $f(x) = \dfrac{px^2 + qx + r}{ax + b}$

By quotient rule.

$$f(x) = \frac{(ax+b)\frac{d}{dx}(px^2+qx+r) - (px^2+qx+r)\frac{d}{dx}(ax+b)}{(ax+b)^2}$$

$$= \frac{(ax+b)(2px+q)-(px^2+qx+r)(a)}{(ax+b)^2}$$

$$= \frac{2apx^2+aqx+2bpx+bq-apx^2-aqx-ar}{(ax+b)^2}$$

$$= \frac{apx^2+2bpx+bq-ar}{(ax+b)^2}$$

(D) Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$$f'(x) = \frac{d}{dx}\left(\frac{a}{x^4}\right) - \frac{d}{dx}\left(\frac{b}{x^2}\right) + \frac{d}{dx}(\cos x)$$

$$= a\frac{d}{dx}(x^{-4}) - b\frac{d}{dx}(x^{-2}) + \frac{d}{dx}(\cos x)$$

$$= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x)$$

$$\left[\frac{d}{dx}(x^n) = nx^{n-1} \text{ and } \frac{d}{dx}(\cos x) = -\sin x\right]$$

$$= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x$$

OBJECTIVE Type Questions

[1 mark]

Multiple Choice Questions

1. The derivation of e^{-9x} is:

(a) e^{-9x} (b) $9e^{-9x}$

(c) $-9e^{-9x}$ (d) none

Ans. *(c)* $-9e^{-9x}$

Explanation: Let $y = e^{-9x}$

$$\frac{d(e^{-9x})}{dx} = e^{-9x} \times \frac{d(-9x)}{dx}$$

$$= -9e^{-9x}$$

2. Derivation of $5e^{\log x}$ is:

(a) $5x^4$ (b) x^5

(c) $4x^4$ (d) none

Ans. *(a)* $5x^4$

Explanation: $\frac{d(5e^{\log x})}{dx} = e^{\log_x 5}$

$$= x^5$$

$$= 5x^4$$

3. Derivation of $\sin^3 x$ is:

(a) $\cos^3 x$ (b) $3\sin^3 x$

(c) $3\sin^2 x \cos x$ (d) 0

Ans. *(c)* $3\sin^2 x \cos x$

Explanation: $\frac{d(\sin^3 x)}{dx} = (\sin x)^3$

$$= 3\sin^2 x \times \frac{d(\sin x)}{dx}$$

$$= 3\sin^2 x \cos x$$

4. $f(x) = x^2 - 5x + 7 f'(3) =$ .... :

(a) 11 (b) 1

(c) 4 (d) 0 [Diksha]

Ans. *(b)* 1

Explanation: Given,

$$f(x) = x^2 - 5x + 7$$

$$f'(x) = 2x - 5$$

$$f'(3) = 2 \times 3 - 5$$

$$= 6 - 5$$

$$= 1$$

5. Derivation of $\log b$ is:

(a) $\frac{1}{b}$ (b) 0

(c) not define (d) 1

Ans. *(b)* 0

Explanation: $\frac{d\,(\log b)}{dx}$

Here, $\log b$ is constant so derivative of constant is 0.

6. Derivation of $\cos 0°$ is:

(a) 0 (b) 1

(c) 2 (d) 3

Ans. *(a)* 0

Explanation: $\frac{d(\cos 0°)}{dx}$

Value of $\cos 0° = 1$, which is a constant

So, the derivative of constant is 0.

7. Derivation of $\left(\frac{x}{2} + \frac{2}{x}\right)$ is:

(a) $\frac{1}{2} + \frac{2}{x^2}$ (b) $\frac{1}{2} - \frac{2}{x^2}$

(c) $\frac{x}{2} - \frac{2}{x^2}$ (d) $\frac{1}{2} - \frac{2}{x}$ [Diksha]

Ans. *(b)* $\frac{1}{2}-\frac{2}{x^2}$

Explanation: $\frac{d}{dx}\left(\frac{x}{2}+\frac{2}{x}\right)$

$$=\frac{d\left(\frac{x}{2}\right)}{dx}+\frac{d\left(\frac{2}{x}\right)}{dx}$$

$$=\frac{1}{2}-\frac{2}{x^2}$$

8. Derivation of $\tan^2 x$ is:

(a) $2\tan x\cdot\sec^2 x$ (b) $\cot^2 x$

(c) 0 (d) None

Ans. *(a)* $2\tan x\cdot\sec^2 x$

Explanation: $\frac{d(\tan^2 x)}{dx}=\frac{d(\tan x)^2}{dx}2$

$$=2\tan x\frac{d(\tan x)}{dx}$$

$$=2\tan x\sec^2 x$$

9. Derivation of ax^3+2x is:

(a) 0 (b) $3ax^2+2$

(c) $3x^2+2$ (d) none

Ans. *(b)* $3ax^2+2$

Explanation: $\frac{d(ax^3+2x)}{dx}=\frac{adx^3}{dx}+\frac{2dx}{dx}$

$$=3ax^2+2$$

10. If $y=(\sin x+\tan x)$, then $\frac{dy}{dx}$ at $x=\frac{\pi}{3}$ is:

(a) $\frac{\sqrt{3}}{2}+\sqrt{3}$ (b) $\sqrt{\frac{3}{2}}+\frac{4}{3}$

(c) $\frac{\sqrt{3}}{2}+4$ (d) $\frac{9}{2}$ [Diksha]

Ans. *(d)* $\frac{9}{2}$

Explanation: Given: $y=(\sin x+\tan x)$

To find: $\frac{dy}{dx}$ at $x=\frac{\pi}{3}$

$$y=(\sin x+\tan x)$$

Differentiation w.r.t. 'x'

$$\frac{dy}{dx}=(\cos x+\sec^2 x)$$

$$\left[\text{As } \frac{d}{dx}(\tan x)=\sec^2 x;\ \frac{d}{dx}(\sin x)=\cos x\right]$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{3}}=\left(\cos\frac{\pi}{3}+\sec^2\frac{\pi}{3}\right)$$

$$=\frac{1}{2}+(2)^2=\frac{1}{2}+4=\frac{9}{2}$$

Assertion Reason Questions

Direction: In the following questions, a statement of Assertion (A) is followed by a statement of Reason (R).

Choose the correct answer out of the following choices.

(a) Both (A) and (R) are true and (R) is the correct explanation of A.

(b) Both (A) and (R) are true but (R) is not the correct explanation of (A).

(c) (A) is true but (R) is false.

(d) (A) is false but (R) is true.

11. Let $u=f(x)$ and $v=g(x)$. Then,

Assertion(A): $(uv)'=u'v+uv'$ is a Leibnitz rule or product rule.

Reason (R): $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v'}$ is a Leibnitz rule or quotient rule.

Ans. *(c) (A) is true but (R) is false.*

Explanation: Let $u=f(x)$ and $v=g(x)$.

Then, $(uv)'=u'v+uv'$

This is referred as Leibnitz rule or the product rule for differentiating product of functions.

Similarly, the quotient rule is $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{v^2}$.

12. Assertion (A): The derivation of $f(x)=x^3$ is x^2.

Reason (R): The derivation of $f(x)=x^n$ is nx^{n-1}.

Ans. *(d) A is false but R is true.*

Explanation: We have, $f(x)=x^3$

We know that the derivation of x^n is nx^{n-1}.

$\Rightarrow$ $f'(x)=3x^2\neq x^2$

13. Assertion (A): The derivation of $y=2x-\frac{3}{4}$ is 2.

Reason (R): The derivation of $y=cx$ is c.

Ans. *(a) Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: We have, $y=2x-\frac{3}{4}$

$\Rightarrow$ $\frac{dy}{dx}=(2\times 1)-0=2$

14. Assertion (A): The derivation of $h(x)=\frac{x+\cos x}{\tan x}$ is $\frac{(1-\sin x)\tan x-(x+\cos x)\sec^2 x}{(\tan x)^2}$.

Reason (R): $\left(\frac{u}{v}\right)'=\frac{u'v-uv'}{(v)^2}$.

Ans. (a) *Both (A) and (R) are true and (R) is the correct explanation of (A).*

Explanation: We have, $h(x) = \frac{x + \cos x}{\tan x}$ ...(i)

Differentiating both sides of (i) w.r.t. 'x', we get

$$h'(x) = \frac{(x + \cos x)' \tan x - (x + \cos x)(\tan x)'}{(\tan x)^2}$$

$$= \frac{(1 - \sin x)\tan x - (x + \cos x)\sec^2 x}{(\tan x)^2}$$

CASE BASED Questions (CBQs)

[**4** & **5** marks]

Read the following passages and answer the questions that follow:

15. *Let f be a real valued function, the function defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ whenever the limit exists is defined to be the derivative of f at x.*

For a function, f(x) = tan x, answer the following questions.

(A) Find the value of $f(x + h) - f(x)$.

(B) Find the value of $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

(C) Find the derivative of $f(x) = x^3$ using the first principle.

Ans. (A) $f(x + h) - f(x) = \tan(x + h) - \tan x$

$$= \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}$$

$$= \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x . \cos(x+h)}$$

$$= \frac{\sin(x+h-x)}{\cos x . \cos(x+h)} = \frac{\sin h}{\cos x \cos(x+h)}$$

(B) $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \lim_{h \to 0} \frac{1}{\cos x \cdot \cos(x+h)}$$

$$= \lim_{h \to 0} \frac{1}{\cos x \cdot \cos(x+h)} \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$= \frac{1}{\cos^2 x}$$

(C) By definition,

$$f'(x) = \lim_{h \to 0} \frac{[f(x+h) - f(x)]}{h}$$

Now, substitute $f(x) = x^3$ in the above equation:

$$f'(x) = \lim_{h \to 0} \frac{[(x+h)^3 - x^3]}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x^3 + h^3 + 3xh(x+h) - x^3)}{h}$$

$$f'(x) = \lim_{h \to 0} (h^3 + 3x(x+h))$$

Substitute $h = 0$, we get

$$f'(x) = 3x^2$$

Therefore, the derivative of the function $f'(x) = x^3$ is $3x^2$.

16. *Let f be a real valued function, the function defined by $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$*

For a function f(x) = sin x + cos x, answer the following questions.

(A) The derivative of f(x) w.r.t x is:
- (a) $\cos x + \sin x$
- (b) $\cos x - \sin x$
- (c) 0
- (d) none of these

(B) The value of $f'(0)$ is:
- (a) 2
- (b) 0
- (c) 1
- (d) –1

(C) The value of $f'(90°)$ is:
- (a) 2
- (b) 0
- (c) 1
- (d) –1

(D) The value of $f'(60°)$ is:
- (a) 2
- (b) 0
- (c) 1
- (d) –1

(E) The derivative of $x^2 \cos x$ is:
- (a) $2x \sin x - x^2 \sin x$
- (b) $2x \cos x - x^2 \sin x$
- (c) $2x \sin x - x^2 \cos x$
- (d) $\cos x - x^2 \sin x \cos x$

Ans: (A) *(b)* $\cos x - \sin x$

Explanation: Since $f(x) = \sin x + \cos x$

Hence derivative, $f'(x) = \cos x - \sin x$

(B) *(c)* 1

Explanation: $f'(x) = \cos x - \sin x$

$$f'(0) = \cos 0° - \sin 0°$$
$$= 1 - 0$$
$$= 1$$

(C) (d) −1

Explanation: $f'(x) = \cos x - \sin x$

$f'(90) = \cos 90° - \sin 90°$

$= 0 - 1 = -1$

(D) (d) −1

Explanation: $f'(x) = \cos x - \sin x$

$f'(60) = \cos 60 - \sin 60$

$= \frac{1}{2} - \frac{3}{2}$

$= -1$

(E) (b) $2x \cos x - x^2 \sin x$

Explanation: $\frac{d}{dx}(x^2 \cos x)$

Using the formula

$$\frac{d}{dx}[f(x)g(x)] = f(x)\left[\frac{d}{dx}g(x)\right] + g(x)\left[\frac{d}{dx}f(x)\right]$$

$$\frac{d}{dx}(x^2 \cos x) = x^2\left[\frac{d}{dx}(\cos x)\right] + \cos x\left[\frac{d}{dx}x^2\right]$$

$= x^2(-\sin x) + \cos x(2x)$

$= 2x \cos x - x^2 \sin x$

VERY SHORT ANSWER Type Questions (VSA)

[1 mark]

17. Differentiate $x^3 + 3^3 + 3^x$ with respect to x. **[Delhi Gov. QB 2022]**

Ans. Let, $f(x) = 3x^x + x^3 + 3^3$

$$f'(x) = \frac{d}{dx}(3^x + x^3 + 3^3)$$

$$= \frac{d}{dx}(3^x) + \frac{d}{dx}(x^3) + \frac{d}{dx}(3^3)$$

$f'(x) = 3 \log 3 + 3x^2$

18. If $y = x.e^{\sin x^2}$ then find $\frac{dy}{dx}$.

Ans. $y = x.e^{\sin x^2}$

$$\frac{dy}{dx} = x\frac{d}{dx}\left(e^{\sin x^2}\right) + \left(e^{\sin x^2}\right)\frac{d}{dx}(x)$$

$$\frac{dy}{dx} = x.e^{\sin x^2} \cdot \cos x^2 \cdot 2x + e^{\sin x^2} \cdot 1$$

$$= 2x^2 \cdot e^{\sin x^2} \cdot \cos x^2 + e^{\sin x^2}.$$

So required solution is

$$\frac{dy}{dx} = e^{\sin x^2}(2x^2 \cos x^2 + 1)$$

19. Differentiate $\sin^2(x^3 + x - 1) + \frac{1}{\sec^2(x^3 + x - 1)}$ with respect to x. **[Delhi Gov. QB 2022]**

Ans. Let $f(x) = \sin^2(x^3 + x - 1) + \frac{1}{\sec^2(x^3 + x - 1)}$

$f(x) = \sin^2(x^3 + x - 1) + \cos^2(x^3 + x - 1)$ $\left[\because \cos^2 x = \frac{1}{\sec^2 x}\right]$

$f(x) = 1$ $\left[\because \sin^2 x + \cos^2 x = 1\right]$

$$f'(x) = \frac{d(1)}{dx}$$

$f'(x) = 0$

20. Differentiate the functions with respect to x: $\frac{x^4 + x^3 + x^2 + 1}{x}$ **[NCERT Exemplar]**

Ans. Let $y = \frac{x^4 + x^3 + x^2 + 1}{x}$

Dividing by x we get

$$\Rightarrow \quad y = x^3 + x^2 + x + \frac{1}{x}$$

Differentiating a given equation with respect to x

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\left(x^3 + x^2 + x + \frac{1}{x}\right)$$

$$= 3x^2 + 2x + 1 - \frac{1}{x^2}$$

21. Find the derivative of tan x°.

Ans. Let $y = \tan x°$

On converting it into radian, we get

$$y = \tan \frac{\pi x}{180°}$$

So,

$$\frac{dy}{dx} = \sec^2 \frac{\pi x}{180} \times \left(\frac{\pi}{180}\right)$$

$$\frac{dy}{dx} = \frac{\pi}{180°} \sec^2 \frac{\pi x}{180°}$$

SHORT ANSWER Type-I Questions (SA-I)

[2 marks]

22. Evaluate $\frac{x^3 + x^2 + x + 1}{x^2}$.

Ans. $\frac{d}{dx}\left(\frac{x^3+x^2+x+1}{x^2}\right) = \frac{d}{dx}\left(x+1+\frac{1}{x}+\frac{1}{x^2}\right)$

$= \frac{d}{dx}x + \frac{d}{dx}\cdot 1 + \frac{d}{dx}\frac{1}{x} + \frac{d}{dx}\left(\frac{1}{x^2}\right)$

$= 1 + 0 - \frac{1}{x^2} + \left(-\frac{2}{x^3}\right)$

$= 1 - \frac{1}{x^2} - \frac{2}{x^3}$

23. Find $(3x + 5)(1 + \tan x)$. **[NCERT Exemplar]**

Ans. Let $y = (3x + 5)(1 + \tan x)$

Applying product rule of differentiation that is

$\Rightarrow \frac{dy}{dx} = (1+\tan x)\frac{d}{dx}(3x+5) + (3x+5)\frac{d}{dx}(1+\tan x)$

$\Rightarrow \frac{dy}{dx} = 3(1 + \tan x) + (3x + 5)\sec^2 x$

$= 3x\sec^2 x + 5\sec^2 x + 3 + 3\tan x.$

(by using product rule)

Hence, the required answer is

$3x\sec^2 x + 5\sec^2 x + 3\tan x + 3.$

24. Differentiate $(3x - 5)(1 + \cot x)$.

Ans. Let $y = (3x - 5)(1 + \cot x)$

$\therefore \frac{dy}{dx} = \frac{d}{dx}[(3x - 5)(1 + \cot x)]$

$= (3x - 5)\frac{d}{dx}(1 + \cot x) + (1 + \cot x)\frac{d}{dx}(3x - 5)$

$= (3x - 5)(-\operatorname{cosec}^2 x) + (1 + \cot x).\ 3$

$= -(3x - 5)\operatorname{cosec}^2 x + 3(1 + \cot x)$

$= -3x\operatorname{cosec}^2 x + 5\operatorname{cosec}^2 x + 3\cot x + 3$

25. Evaluate $\frac{x^5 - \cos x}{\sin x}$. **[NCERT Exemplar]**

Ans. Given $y = \frac{x^5 - \cos x}{\sin x}$

$\frac{d}{dx}(x^5 - \cos x)/\sin x = \left[\sin x.\frac{d}{dx}(x^5 - \cos x) - (x^5 - \cos x).\frac{d}{dx}(\sin x)\right] / \sin^2 x$

By using quotient rule,

$= \frac{[\sin x(5x^4 + \sin x) - (x^5 - \cos x)(\cos x)]}{\sin^2 x}$

$= \frac{[5x^4.\sin x + \sin^2 x - x^5\cos x + \cos^2 x]}{\sin^2 x}$

$= \frac{[5x^4\sin x - x^5\cos x + (\sin^2 + \cos^2 x)]}{\sin^2 x}$

$= \frac{[5x^4\sin x - x^5\cos x + 1]}{\sin^2 x}$

Hence, the required answer is

$\frac{[5x^4\sin x - x^5\cos x + 1]}{\sin^2 x}.$

SHORT ANSWER Type-II Questions (SA-II)

[3 marks]

26. Find the derivative of $\frac{3x + 4}{5x^2 - 7x + 9}$.

[NCERT Exemplar]

Ans. Given $\frac{3x+4}{5x^2 - 7x + 9}$

Applying quotient rule of differentiation that is

$\Rightarrow \frac{d}{dx}\left(\frac{t}{s}\right) = \frac{s\cdot\frac{dt}{dx} - t\cdot\frac{ds}{dx}}{s^2}$

$\Rightarrow \quad y = \frac{3x+4}{5x^2 - 7x + 9}$

On applying the rule, we get

$$(5x^2-7x+9)\frac{d}{dx}(3x+4)-(3x+4)$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{d}{dx}(5x^2-7x+9)}{(5x^2-7x+9)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{3(5x^2-7x+9)-(3x+4)(10x-7)}{(5x^2-7x+9)^2}$$

On differentiation, we get

$$=\frac{(15x^2-21x+27-30x^2+21x-40x+28)}{(5x^2-7x+9)^2}$$

$$=\frac{(-15x^2-40x+55)}{(5x^2-7x+9)^2}$$

$$=\frac{(55-40x-15x^2)}{(5x^2-7x+9)^2}$$

27. **Find the derivation of $\sqrt{\frac{1-\cos x}{1+\cos x}}$.**

Ans. Let $\sqrt{\frac{1-\cos x}{1+\cos x}}$

$$y=\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}$$

Here, $1-\cos x = 2\sin^2\frac{x}{2}$

$1+\cos x = 2\cos^2\frac{x}{2}$

So, $y=\sqrt{\tan^2\frac{x}{2}}$

$$y=\tan\frac{x}{2}$$

$$\frac{dy}{dx}=\sec^2\frac{x}{2}\cdot\left(\frac{1}{2}\right).$$

28. **Find the derivative of $\frac{a+b\sin x}{c+d\cos x}$.** **[NCERT Exemplar]**

Ans. Given $y=\frac{a+b\sin x}{c+d\cos x}$

Applying quotient rule of differentiation that is

$$\Rightarrow \frac{d}{dx}\left(\frac{t}{sy}\right)=\frac{sy\cdot\frac{dt}{dx}-t\cdot\frac{dsy}{dx}}{sy^2}$$

$$\Rightarrow \quad y=\frac{a+b\sin x}{c+d\cos x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{(c+d\cos x)\frac{d}{dx}(a+b\sin x)-(a+b\sin x)\frac{d}{dx}(c+d\cos x)}{(c+d\cos x)^2}$$

On differentiating, we get

$$\Rightarrow \frac{dy}{dx}=\frac{(c+d\cos x)(b\cos x)-(a+b\sin x)(-d\sin x)}{(c+d\cos x)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{b\cos x(c+d\cos x)+d\sin x(a+b\sin x)}{(c+d\cos x)^2}$$

$$=\frac{[cb\cos x+bd\cos^2 x+ad\sin x+bd\sin^2 x]}{(c+d\cos x)^2}$$

$$=\frac{[cb\cos x+ad\sin x+bd(\cos^2 x+\sin^2 x)]}{(c+d\cos x)^2}$$

$$=\frac{cb\cos x+ad\sin x+bd}{(c+d\cos x)^2}$$

29. **If $y=\cos x\cdot e^{\sin x^2}$ then find $\frac{dy}{dx}$.**

Ans. Given, $y = \cos x \,.\, e^{\sin x^2}$

Using product rule.

$$=\cos x\frac{d}{dx}(e^{\sin x^2})+(e^{\sin x^2})\frac{d}{dx}(\cos x)$$

$$\frac{dy}{dx}=\cos x\cdot e^{\sin x^2}\cdot\cos x^2\cdot 2x-e^{\sin x^2}\cdot\sin x$$

$$=2x\cdot\cos x e^{\sin x^2}\cdot\cos x^2-e^{\sin x^2}\cdot\sin x$$

So required solution is,

$$\frac{dy}{dx}=e^{\sin x^2}(2x\cos x\cdot\cos x^2-\sin x)$$

LONG ANSWER Type Questions (LA)

[4 & 5 marks]

30. **Find derivative of $(2x-7)^2(3x+5)^3$ with respect to x.** **[NCERT Exemplar]**

Ans. Given $y=(2x-7)^2(3x+5)^3$

Applying product rule of differentiation that is

$$\Rightarrow \frac{d}{dx}(t.y)=y\cdot\frac{dt}{dx}+t\cdot\frac{dy}{dx}$$

$\Rightarrow \quad y = (2x-7)^2 (3x+5)^3$

$$\Rightarrow \frac{dy}{dx} = (3x+5)^3 \frac{d}{dx}(2x-7)^2 + (2x-7)^2 \frac{d}{dx}(3x+5)^3$$

On differentiating, we get

$$\Rightarrow \frac{dy}{dx} = (2)(3x+5)^3 2(2x-7)^1 + (3)(2x-7)^2 3(3x+5)^2$$

$$\Rightarrow \frac{dy}{dx} = 4(3x+5)^3(2x-7) + 9(2x-7)^2(3x+5)^2$$

On simplification, we get

$$\Rightarrow \frac{dy}{dx} = (2x-7)(3x+5)^2[4(3x+5)+9(2x-7)]$$

$$\Rightarrow \frac{dy}{dx} = (2x-7)(3x+5)^2(30x-43)$$

31. Find derivative of $\sin^2 x.\cos x$ with respect to $e^{\sin(e^x)}$.

Ans. Let $u = \sin^2 x \cdot \cos x$

And $v = e^{\sin(e^x)}$

Now we have to find $\frac{du}{dv}$.

So, $\frac{du}{dx} = d(\sin^2 x \cdot \cos x)$

$$= \sin^2 x \frac{d(\cos x)}{dx} + \cos x \frac{d(\sin^2 x)}{dx}$$

$= \sin^2 x(-\sin x) + \cos x . 2 \sin x \cos x$

$= -\sin^3 x + 2 \sin x \cos^2 x$

$= \sin x \,(2 \cos^2 x - \sin^2 x)$

$= \sin x \,(\cos^2 x + \cos^2 x - \sin^2 x)$

$= \sin x \,(\cos^2 x + \cos 2x)$

and $\frac{dv}{dx} = \frac{d}{dx}(e^{\sin(e^x)})$

$= e^{\sin(e^x)} \cdot \cos(e^x) \cdot e^x$

Now, $$\frac{du}{dv} = \frac{\sin x(\cos^2 x + \cos 2x)}{e^{\sin(e^x)} \cdot \cos e^x \cdot e^x}$$

32. Differentiate with respect to x using the first principle $x^2 \cos x$. **[Diksha]**

Ans. Given, $f(x) = x^2 \cos x$

First principle, $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 \cos(x+h) - x^2 \cos x}{h}$$

Using L-Hospital's rule, we have,

$$f'(x) = \lim_{h \to 0} \frac{2(x+h)\cos(x+h) + (x+h)^2 [-\sin(x+h)] - 0}{1}$$

$$f'(x) = \lim_{h \to 0} 2(x+h)\cos(x+h) + (x+h)^2 [-\sin(x+h)]$$

$f'(x) = 2(x+0)\cos(x+0) + (x+0)^2 [-\sin(x+0)]$

$f'(x) = 2x \cos x + x^2 [-\sin x]$

$\therefore f'(x) = 2x \cos x - x^2 \sin x$